JOURNAL OF APPLIED POLYMER SCIENCE VOL. 20, 891–901 (1976)

Theoretical Model for the Elastic Behavior of Composites Reinforced with Short Fibers

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Synopsis

The two simplest models that can be put forward to account for the elasticity of composite materials are the Reuss model and the Voigt model in which the constituents undergo, respectively, the same stress or the same strain. Experimental measurements always fall between the values predicted by these models. We propose correcting the Reuss model by stating

$$\sigma_f = K\sigma_m$$

σ_f and σ_m being the average stresses undergone, respectively, by the reinforcing agent and the matrix. Similarly, we shall modify the Voigt model by supposing

$$\epsilon_f = L\epsilon_m$$

ϵ_f and ϵ_m being the average strains undergone, respectively, by the reinforcing agent and the matrix. K and L are interrelated tensors which depend on the nature of the reinforcing agent, on its possible orientation, and on the mechanical behavior of the interface and also on the moduli of the constituents. We have developed the equations for determining the tensors with regard to fiber composite, taking into account the characteristics of the fibers (length, diameter, orientation, interface). The evaluation of K and L enables us, therefore, to calculate the modulus or the compliance. Conversely, by measuring the modulus or the compliance, one can determine K or L and, in this way, obtain data on the mechanism of load transfer from the matrix to the reinforcing agent and thus on the behavior of their interface. The theoretical values of the Young modulus calculated from our model are in good agreement with the experimental values obtained by Lees.[8]

INTRODUCTION

Short fibers are a frequently used reinforcing agent for thermoplastics, and yet there are few theoretical models that enable us to forecast satisfactorily the modulus of elasticity of such a composite.

In composites with dispersed fibers, stresses are transferred from the matrix to the fibers, mainly by shear. The distribution of stresses along fibers has been studied by numerous authors, both for elastic and for plastic deformations, or for a combination of the two.[1–4]

Another way of viewing the problem is to consider that the composite is macroscopically homogeneous and generally speaking anisotropic. In this case, one defines average stresses and strains. The relationships between the values depend not only on the macroscopic deformations of the composite but also on micromechanics.

The simplest models are those which make use of the Reuss and the Voigt averages. If the strains are uniform, the modulus of the composite is given by (Voigt model)

$$E = E_f v_f + E_m v_m.$$

If the stresses are uniform (Reuss model), then the modulus is

$$E = \frac{E_m E_f}{E_m v_f + E_f v_m}.$$

It is possible to show by energetistic considerations that the true modulus of a composite falls between the two values if no slip takes place at the interface.[5] Working from these two averages, Ward has formulated a model which makes it possible to take into account the orientation of the fibers but in which the length of the fibers is not taken into consideration.[6]

We have determined the moduli of elasticity of composites with short fibers oriented in various ways by using the relations in such a form that it was easy to make the correspondence with the results given by micromechanics.

THEORETICAL MODEL

The stresses applied to the composite are transferred to the fibers by the matrix, and so it is possible to express the average stresses and strains of the fibers in terms of those of the matrix. An intermediate behavior pattern between those of the models of Voigt and Reuss can be expressed as follows:

$$\sigma_f = K\sigma_m \tag{1}$$

$$\epsilon_f = L\epsilon_m \tag{2}$$

in which K and L are tensors of the fourth order. If the composite is deformed elastically, each phase is also deformed elastically, and one must find relationships between average stresses and strains:

$$\epsilon_f = S_f\sigma_f \tag{3}$$

$$\epsilon_m = S_m\sigma_m. \tag{4}$$

With the introduction of these relationships into (1) and (2), we obtain

$$LS_m = S_f K$$

and, generally speaking, the tensors L and K can each be expressed as a function of the other:

$$L = S_f K S_m^{-1}. \tag{5}$$

The modulus of the composite can be determined in the following way.

The average stress and the average strain are, respectively,

$$\sigma = v_f\sigma_f + v_m\sigma_m \tag{6}$$

$$\epsilon = v_f\epsilon_f + v_m\epsilon_m. \tag{7}$$

By introducing (1) and (2), these relations enable us to obtain the compliance tensor S:

$$S = (v_fL + v_mI)S_m(v_fK + v_mI)^{-1} \tag{8}$$

We return to the Reuss expression in assuming that $K = I$, whence, by means of (5), $L = S_fS_m^{-1}$, and therefore

$$S = v_fS_f + v_mS_m$$

We come back to Voigt's expression in supposing $K = S_f^{-1}S_m$, whence, by means of (5), $L = I$, and so

$$S = S_fS_m(v_fS_m + v_mS_f)^{-1}$$

In the case where some slip of the fibers takes place, we must add the extra deformation that it causes to the deformations due to the matrix and to the fibers, and (7) becomes

$$\epsilon = v_f\epsilon_f + v_m\epsilon_m + \epsilon_g.$$

If the relation between the slip and the strain can be represented by the tensor M defined by

$$M = \epsilon_g\epsilon_m^{-1}$$

that is to say, if we can invert ϵ_m, we can then obtain a relation identical to (8) by replacing L in this relation by

$$L' = L + \frac{1}{v_f} M.$$

In the opposite case, we must retain the term ϵ_g, and the compliance tensor of the composite is

$$S_m = (v_fK + v_mI)S_m(v_fL + v_mI)^{-1} + \epsilon_g[(v_fK + v_mI)\sigma_m]^{-1}.$$

If the fiber–matrix bond is weak, one can allow that the areas from which the matrix is absent are deformed like the matrix, one part of the deformation being due to the slip and another to the deformation of the fibers. All this amounts to saying that

$$L' = I \qquad (\text{with } K \neq S_f^{-1}S_m). \tag{9}$$

Relations (3) to (8) are general, and the tensors K and L represent the transfer of the stresses due as much to the shape of the elements of the dispersed phase as to their orientation. In the case of fibers characterized by one preponderent dimension, their length, it is important to dissociate the influence of the orientation from that of the other factors affecting the reinforcement. We can admit that a fiber only reinforces in one direction and that its deformation is proportional to the uniaxial deformation of the matrix in the direction of the fiber. A fiber whose orientation is determined by the

Euler angles θ and ϕ can therefore be characterized by the parameter l defined by

$$\epsilon_f(\theta,\phi) = l\epsilon_m(\theta,\phi)$$

in which $\epsilon_f(\theta,\phi)$ and $\epsilon_m(\theta,\phi)$ are the uniaxial strains in the direction of the fiber.

Leaving aside transverse reinforcement, the compliance tensor of the fibers is reduced to

$$\sigma_f(\theta,\phi) = E_f\epsilon_f(\theta,\phi).$$

We then have

$$\sigma_f(\theta,\phi) = \frac{lE_f}{E_m}\sigma_m(\theta,\phi) = k\sigma_m(\theta,\phi).$$

We can express $\sigma_f(\theta,\phi)$ in terms of the state of stress of the matrix:

$$\sigma_f(\theta,\phi) = kE_mA(\theta,\phi)S_m\sigma_m$$

in which A is a tensor of the fourth order whose components are the products of the director cosines $a_i(\theta,\phi)$ of the direction (θ,ϕ):

$$A_{ijkl} = a_ia_ja_ka_l.$$

The stress is determined by calculating an integral in all the directions of the fibers. Let $f(\theta,\phi)$ be the volumic fraction of fibers in the direction (θ,ϕ) per unit of solid angle, so that

$$\int_0^{\pi/2}\int_0^{2\pi} f(\theta,\phi)\, d\theta d\phi = 1.$$

We then have

$$\sigma = \int_0^{\pi/2}\int_0^{2\pi} f(\theta,\phi) \sin\theta\; A(\theta,\phi)S_m\, d\theta\, d\phi.$$

ϵ is determined in a similar manner, and by stating then

$$K = \int_0^{\pi/2}\int_0^{2\pi} E_mk\, f(\theta,\phi) \sin\theta\; A(\theta,\phi)S_m\, d\theta\, d\phi \tag{10}$$

and

$$L = \frac{1}{E_f}KS_m^{-1} \tag{11}$$

we obtain a relation identical to eq. (8).

If there is some slip of the fibers, the latter must be characterized by the two parameters k and l generally depending on the deformation and therefore also on the direction; S is then given by

$$S = (v_fL' + v_mI)S_m(E_mv_fKS_m + v_mI)^{-1} \tag{12}$$

in which K and L' must be evaluated separately; K is always given by relation (10) in which, this time, k depends on the direction. L' can be replaced by I in the same circumstances as previously, eq. (9).

It is the parameters k and l that can be determined by micromechanics; indeed, they are proportional, respectively, to the average stress and the average strain in the fibers. So, if we consider an elastic composite, with fibers of length L and radius r_1, we can use the model of Cox[1] which gives the mean stress in the fibers:

$$k\sigma_m = \bar{\sigma}_f = \sigma_m \frac{E_f}{E_m}\left(1 - \frac{\tanh \beta\left(\frac{L}{2} - x\right)}{\beta \frac{L}{2}}\right) \tag{13}$$

with

$$\beta = \left(\frac{H}{E_f A}\right)^{1/2}$$

and

$$H = \frac{2\pi G}{\log_e\left(\frac{r_1}{r_0}\right)}$$

where G is the shear modulus of the matrix, r_0 is the average distance between fibers, and A is the cross section of the fibers. This enables us to compute k from the characteristics of the fibers and of the matrix; l is given by

$$l = \frac{E_m}{E_f} k.$$

Different authors assume other hypotheses: more sophisticated theories on the elastic deformation, plastic deformation of the matrix around the fibers, or a combination of the two can be taken into account.[4,9,10,11] In all these cases, we can compute the value of σ_f along the fibers, and the value of k is given by

$$k = \frac{\bar{\sigma}_f}{\sigma_m}$$

$\bar{\sigma}_f$ being the mean value of σ_f along the fibers. If the fibers are not of identical length and diameter, it is sufficient to calculate the average values of k and l; for example,

$$\bar{k} = \int_0^\infty \int_0^\infty k(L,D)\, f(L,D)\; dL\, dD \tag{14}$$

in which $f(L,D)$ is the volumic fraction of fibers of length L and of diameter D, so that

$$\int_0^\infty \int_0^\infty f(L,D)\, dL\, dD = 1$$

Let us now develop expression (8) when the matrix is isotropic and the fibers are regularly oriented in all directions. The integrals (11) lead to the following tensor L:

$$L = \frac{E_m}{E_f} k \begin{matrix} 1/5 & 1/15 & 1/15 & 0 & 0 & 0 \\ 1/15 & 1/5 & 1/15 & 0 & 0 & 0 \\ 1/15 & 1/15 & 1/5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/15 \end{matrix}$$

(The indices 1, 2, 3, 4, 5, 6 correspond, respectively, to the pairs of indices of the stress tensors 11, 22, 33, 23, 13, 12).

The tensor S_m is:

$$S_m = \begin{matrix} 1/E_m & -\mu/E_m & -\mu/E_m & 0 & 0 & 0 \\ -\mu/E_m & 1/E_m & -\mu/E_m & 0 & 0 & 0 \\ -\mu/E_m & -\mu/E_m & 1/E_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu)/E_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu)/E_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu)/E_m \end{matrix}$$

By introducing these values into eq. (8), we then obtain a symmetrical tensor S like the tensors K and S_m whose nonzero components are presented in eqs. (15)–(17) [p. 897]

The general application of the foregoing to viscoelastic matrixes can be made fairly easily: the tensor S_m is dependent upon time, as are k and l. In expression (8), therefore, we must replace the tensors by functions of time. The elastic relationship (5) is obviously no longer valid, and it will be necessary to determine both K and L.

DISCUSSION

Elasticity equations cannot be solved with exactitude for a material as complex as a composite with short fibers. Calculated moduli of elasticity therefore always contain approximations.

A first point that should be emphasized is that deformations must be small; indeed, the tensors K and L represent and measure a linear transfer of stress. We have, therefore, ignored terms of the second order. Moreover, the orientation of the fibers has been assumed to be independent of the deformations (10), which is no longer true in the case of large deformations.

We have subsequently left out of consideration the transverse reinforcement of the fibers in expression (10) which is certainly not admissible in the case of an impregnated mat for directions perpendicular to its plane, nor for directions perpendicular to the fibers when these are all parallel.

Furthermore, the model enables us to obtain exact values of the modulus only if we know k and l with accuracy and these depend on the behavior of

$$s_{11} = s_{22} = s_{33} = \frac{(kv_f)^3(20 - 60\mu^2 - 40\mu^3) + (kv_f)^2(15v_m)(16 - 8\mu - 24\mu^2 + 8E_f/E_m - 24\mu^2 E_f/E_m - 16\mu^3 E_f/E_m) + kv_f(15v_m)^2(3 - 2\mu + 6E_f/E_m - 6\mu E_f/E_m + 8\mu^2 E_f/E_m) + (15v_m)^3 E_f/E_m}{E_f[(kv_f)^3(20 - 60\mu^2 - 40\mu^3) + (kv_f)^2(15v_m)(24 - 12\mu - 36\mu^2) + kv_f(15v_m)^2(9 - 6\mu) + (15v_m)^3]} \tag{15}$$

$$s_{12} = s_{13} = s_{23} = \frac{(kv_f)^2(15v_m)(-1 + 8\mu - 16\mu^2 - 2E_f/E_m + 6\mu^2 E_f/E_m + 4\mu^3 E_f/E_m) + kv_f(15v_m)^2(1 - 4\mu - E_f/E_m + 2\mu E_f/E_m - 2\mu E_f/E_m) - \mu(15v_m)^3 E_f/E_m}{E_f[(kv_f)^3(20 - 60\mu^2 - 40\mu^3) + (kv_f)^2(15v_m)(24 - 12\mu - 36\mu^2) + kv_f(15v_m)^2(9 - 6\mu) + (15v_m)^3]} \tag{16}$$

$$s_{44} \doteq s_{55} = s_{66} = \frac{2(1 + \mu)(kv_f) + 15v_m E_f/E_m}{E_f[2(1 + \mu)(kv_f) + (15v_m)]}. \tag{17}$$

TABLE I
Distribution of the Length of the Fibers[a]

	% of total volume of fibers					
Length, cm	1.7%[b]	3.86%	8.3%	13.4%	19.4%	26.5%
0.508	36.2	27.9	17.8	6.3	3	1.2
0.190	17.3	18.9	17.3	12.7	9.7	2.5
0.102	22.1	26.6	33.4	39.9	13.9	9
0.0635	13.9	14.6	19	21.8	47.2	40
0.0381	8.3	9	10.3	14.5	20.7	36.7
0.019	1.2	2.2	1.6	3.5	3.7	6.3
0.0076	1	0.8	0.7	1.3	1.9	4.3
Average length	0.249	0.216	0.190	0.112	0.079	0.051

[a] According to Lees.[8]
[b] Volume fraction of the fibers, in %.

the fibers. In fact, it is better to determine k from experimental modulus measurements and to consider this value as the actual efficacity of the reinforcement. This value of k has a physical meaning (ratio of stress in fibers and in matrix). The "fiber efficiencies" ϕ (defined by $E_c = \phi E_f v_f + E_m V_m$ or in a similar manner) is quite arbitrary. The k parameter varies from 0 to E_f/E_m; so, to compare different composites, it can be more useful to use the

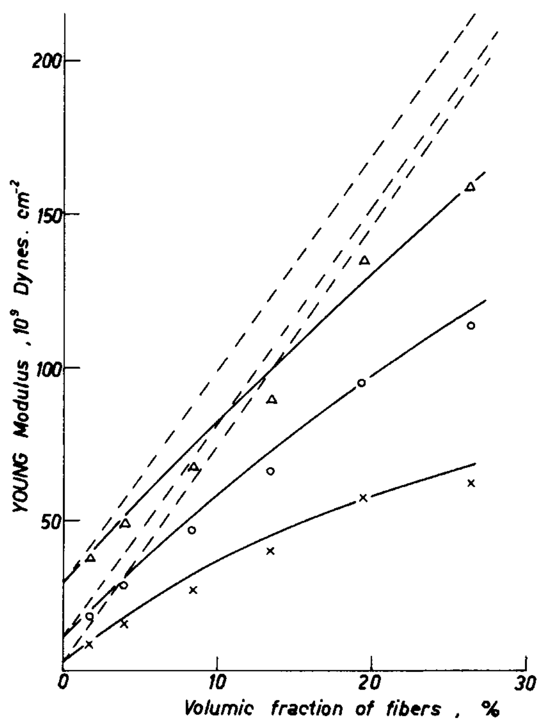

Fig. 1. Experimental and theoretical Young moduli of unidirectional composite vs volumic fraction of fibers for three moduli of the matrix: (x) $E_m = 3.28 \times 10^9$ dynes/cm^2; (O) $E_m = 11.8 \times 10^9$ dynes/cm^2; (Δ) $E_m = 29.6 \times 10^9$ dynes/cm^2. Full lines: theoretical predictions by our model; dotted lines: law of mixtures. Young modulus of the fibers: 716×10^9 dynes/cm^2.

TABLE II
Polyethylene Reinforced with Short Glass Fibers[a]

Young modulus of the matrix, 10^9 dynes/cm^2	Fiber content, %	Young modulus of the composite,[b] 10^9 dynes/cm^2			$kE_mE_f^{-1}$ exper.[c]	$kE_mE_f^{-1}$ calcd.[d]
		A	B	C		
3.28	1.7	15.4	8.96	9.52	0.464	0.523
3.28	3.86	30.8	15.2	17.1	0.423	0.513
3.28	8.3	62.4	27	32.0	0.380	0.480
3.28	13.4	98.8	39.8	45.0	0.349	0.426
3.28	19.4	142	57.7	56.7	0.343	0.331
3.28	26.5	192	62	67.5	0.249	0.300
10.1	1.7	22.1	14.9	18.0	0.393	0.662
10.1	3.86	37.4	27	28.0	0.610	0.664
10.1	8.3	68.7	41.1	48.4	0.507	0.657
10.1	13.4	105	54.7	68.7	0.435	0.604
10.1	19.4	147	86.9	89.8	0.507	0.540
10.1	26.5	197	106	112	0.436	0.498
11.8	1.7	23.8	18.9	19.9	0.590	0.680
11.8	3.86	39.0	28.9	30.1	0.620	0.682
11.8	8.3	70.3	47	51.1	0.581	0.676
11.8	13.4	106	66.8	72.3	0.548	0.623
11.8	19.4	148	95	94.7	0.557	0.566
11.8	26.5	198	114	118	0.471	0.517
15.7	1.7	27.6	21.4	24.1	0.478	0.728
15.7	3.86	42.7	29.2	34.7	0.491	0.713
15.7	8.3	73.8	47.1	56.9	0.519	0.702
15.7	13.4	110	68.6	79.5	0.528	0.667
15.7	19.4	152	96.5	104	0.542	0.625
15.7	26.5	201	115	130	0.458	0.570
19	1.7	30.9	30.7	27.7	0.985	0.744
19	3.86	45.9	35.8	38.6	0.614	0.726
19	8.3	76.9	57.2	61.4	0.640	0.730
19	13.4	112	77.2	84.9	0.588	0.705
19	19.4	154	117	111	0.679	0.639
29.6	1.7	41.2	37.9	38.6	0.709	0.785
29.6	3.86	56.1	49.1	50.1	0.729	0.780
29.6	8.3	86.6	67.6	74.0	0.648	0.781
29.6	13.4	122	89.6	99.2	0.619	0.744
29.6	19.4	163	135	128	0.754	0.712
29.6	26.5	212	159	159	0.644	0.667

[a] The Young modulus of the fibers is 716×10^9 dynes/cm^2, and the diameter is 0.0157 cm (248 filaments of 10 μm). The lengths are taken from Table I.

[b] A = Law of mixtures; B = according to Lees; C = from our model.

[c] Deduced from the experimental Young modulus of the composite.

[d] Calculated from relation (13). The stress distribution along the fibers is taken from the Cox model.[1]

value of kE_m/E_f, which represents the transfer of stresses relative to the maximum transfer.

We have applied the theoretical results to the experimental values determined by Lees.[7,8] He measured the modulus of elasticity of polyethylenes of different moduli reinforced with variable quantities of E glass fibers. The lengths of the fibers were measured after compounding (Table I).

In Table II, we have compared the reinforcement determined experimen-

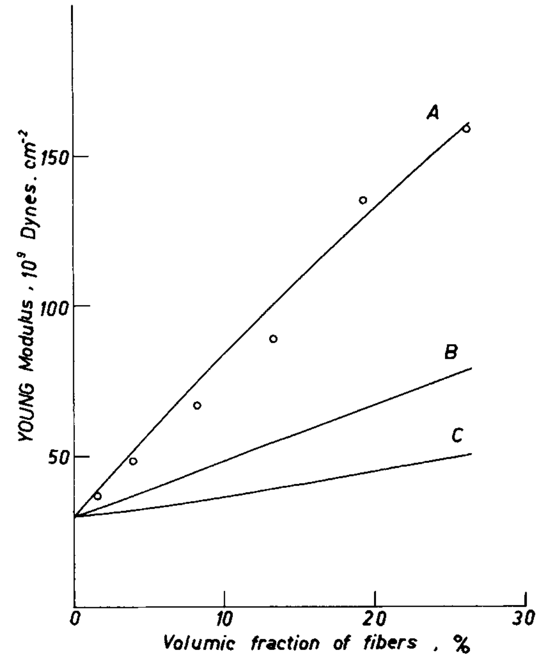

Fig. 2. Predicted Young moduli of three composites with different orientations of fibers: (A) unidirectional [(O) experimental]; (B) planar mat; (C) isotropic composite. Modulus of the matrix: 29.6×10^9 dynes/cm^2; modulus of the fibers: 716×10^9 dynes/cm^2.

tally with the theoretical reinforcement calculated taking for granted elastic deformation and a distribution of tensile stresses along the fiber identical to that of the Cox model.[1]

Values of k are calculated using relations (13) and (14) with the fiber lengths experimentally determined by Lees.[7] Some values of Table II have been plotted on Figure 1 to compare the experimental moduli to the moduli predicted by our theory and the law of mixtures.

The experimental values are much closer to the values of our model than to the law of mixtures. Although the discrepancies are of the order of experimental errors, the experimental values are perhaps somewhat lower than anticipated. This discrepancy may result from plastic deformation of the matrix or nonlinear behavior at the ends of the fibers, or also from imperfect orientation or dispersion of the reinforcing agent. All these factors lead to a lower value of the Young modulus of the composite, which is effectively observed.

The main interest of short fibers is for isotropic reinforcement (or nearly isotropic) and our model enables us to extrapolate the Young modulus of a composite with parallel orientation to the modulus of a composite with another one. Using expression (14), we have shown, in Figure 2, the calculated reinforcements one would expect for a matrix with a Young modulus of 26.9 10^9 dynes/cm^2, for three types of orientation (isotropic, planar, and parallel). The values of k and l, calculated according to expression (13), are identical for the three orientations.

CONCLUSIONS

We are proposing a deformation model for composites reinforced with short fibers. The components of the compliance tensor of the composite have been determined as functions of the parameters k and l, which represent and measure the linear transfer of the stresses in the composite and which are directly linked to the average stresses and strains in the fibers. We have developed these expressions in the case of an isotropic or planar orientation of the fibers. Comparison of the theory with the experiment enables us to obtain *in situ* a quantitative measurement of the transfer of stresses for short fibers oriented in any given way. The parameters k and l used in the expressions of the compliance tensor are directly dependent upon the transfer of the stresses; the macroscopic measurements thus offer access to the properties of the interface, especially important when short fibers are used.

SUMMARY

The two simplest models that can be put forward to account for the elasticity of composite materials are the Reuss model, if we accept that the different constituents undergo the same stress, and the Voigt model, in which the constituents are subjected to identical strains. Experimental measurements always fall between the values predicted by these models. We propose correcting the Reuss model by stating

$$\sigma_f = K\sigma_m$$

σ_f and σ_m being the average stresses undergone, respectively, by the reinforcing agent and the matrix.

Similarly, we shall modify the Voigt model by supposing

$$\epsilon_f = L\epsilon_m$$

ϵ_f and ϵ_m being the average strains undergone, respectively, by the reinforcing agent and the matrix. K and L are the interrelated constants which depend on the nature of the reinforcing agent, on its possible orientation, and on the mechanical behavior of the interface and also on the moduli of these constituents.

We have developed the equations for determining the values of the tensors K and L with regard to fiber-reinforced composites taking into account the following characteristics: length and diameter of the fibers; mechanical behavior of the matrix–fiber interface (elastic, plastic, or stress transfer due to normal forces); orientation of the fibers (we have considered the following three cases: axial, planar, and isotropic orientation); and stress applied to the composite.

The evaluation of K or L enables us, therefore, to calculate the modulus or the compliance. Conversely, by measuring the modulus or the compliance, one can determine K or L and in this way obtain data on the mechanism of load transfer from the matrix to the reinforcing agent and thus on the behavior of their interface. The theoretical values of the Young modulus calculated from our model are in good agreement with the experimental values obtained by Lees.[8]

The authors wish to express their thanks to Professor J. P. Mercier for his encouragement and helpful discussion during this work.

Notations

The indices f and m refer, respectively, to the fibers and to the matrix; the symbols without indices refer to the composite.

E Young's modulus

μ Poisson's ratio

σ, ϵ tensors of stresses and deformations

S compliance tensor

K, L tensors of the fourth order representing the transfer of the stresses and strains.

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Received January 30, 1975
Revised June 2, 1975